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On matching extensions with prescribed and proscribed edge sets II

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Abstract

Let G be a graph with at least $2(m+n+1)$ vertices. Then G is $E(m,n)$ if for each pair of disjoint matchings $M,N \subseteq E(G)$ of size m and n, respectively, there exists a perfect matching F in G such that $M \subseteq F$ and $F \cap N = \emptyset$. In this paper, we extend previous results due to Chen (Discrete Math., to appear) as well as results of the present authors (Aldred et al., Discrete Math., to appear) concerning the property $E(m,n)$. The first extends a result on claw-free graphs and the second generalizes a result about bipartite graphs. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

In this paper all graphs will be finite and, unless otherwise specified, simple as well. Let G be a graph with at least $2(m+n+1)$ vertices. G is said to be $E(m,n)$ if for every pair of disjoint matchings $M,N \subseteq E(G)$ of size m and n respectively, there is a perfect matching F in G such that $M \subseteq F$ and $F \cap N = \emptyset$. If G is $E(n,0)$ we say that G is *n-extendable*. In fact, it was the concept of n-extendability which subsequently gave rise to the property $E(m,n)$. Graphs which are n-extendable have been studied quite extensively (see [10,12]). Some of the early results on this family of graphs are also to be found in the book [6] where their connection with other areas of matching theory are also discussed. For further information on n-extendable graphs, we refer the interested reader to these three sources and the reference lists contained therein.

The first paper to treat the more general concept of $E(m,n)$ was due to Porteous and one of the present authors [14]. In this paper the general theme is the study of when the implication $E(m,n) \to E(p,q)$ does and does not hold. Although it has long been

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known that n-extendability implies $(n-1)$-extendability [7], there are a few surprises in the implication lattice for the property $E(m,n)$. For example, although $E(m,n)$ implies $E(m-1,n)$ for all $m \geqslant 1$, $E(m,n)$ does not always imply $E(m,n-1)$.

2. A result for $K_{1,r}$-free graphs

Chen [2], and independently one of the present authors [13], have proved the following theorem.

Theorem 2.1. *Let $m \geqslant 1$, $r \geqslant 2$ and let G be a $(2m+r-2)$-connected $K_{1,r}$-free graph of even order at least $2m+2$. Then G is $E(m,0)$.*

We extend this result in the theorem below.

Theorem 2.2. *Let m,n and r be non-negative integers with $m \geqslant 1$, $r \geqslant 3$ and let G be a $(2m+n+r-2)$-connected $K_{1,r}$-free graph of even order at least $2m+2n+2$. Then G is $E(m,n)$.*

Proof. The proof is by induction on n. When $n=0$, the result is true by Theorem 2.1. Suppose k is the smallest integer such that there exists a graph G with $|V(G)| \geqslant 2m+2k+2$ which is $(2m+k+r-2)$-connected, even and $K_{1,r}$-free, but not $E(m,k)$. Hence $k \geqslant 1$, and there exist matchings $M=\{e_1,\ldots,e_m\}$ and $K=\{f_1,\ldots,f_k\}$ such that graph $G'=G-V(M)-K$ has no perfect matching. Thus by Tutte's theorem on perfect matchings, and since G is even, there exists a set $S' \subseteq V(G')$ such that $c_0(G'-S') \geqslant |S'|+2$.

By the inductive hypothesis, G is $E(m,k-1)$ and thus each f_i joins two different odd components of $G'-S'$ and so $c_0(G'-S')=|S'|+2$. Denote these odd components by $C_1,\ldots,C_{|S'|+2}$.

In the graph G shrink the subgraphs corresponding to C_i, $i=1,\ldots,|S'|+2$, each to a single vertex, v_i, to form a new graph G'' where we suppress any parallel edges or loops thus formed. Let N denote the number of edges in G'' joining the vertices of $S' \cup V(M)$ to the $|S'|+2$ different v_i's.

Claim. $|S'| \geqslant k$ *(and thus $S' \cup V(M)$ is a cutset in G).*

Suppose, to the contrary, that $|S'| \leqslant k-1$. Then $G-V(M)-S'$ is connected. (For we have removed a set of size $2m+|S'| \leqslant 2m+k-1$ and G is at least $(2m+k+1)$-connected by hypothesis.) But then if $G'-S'$ has an even component, $G'-S'$ is disconnected and hence $G-V(M)-S'$ is disconnected, a contradiction. Thus, there are no even components in $G'-S'$.

Let $D=\{v_i\colon 1 \leqslant i \leqslant |S'|+2\}$. Now adding the k edges $f_1,\ldots,f_k$ to the graph D yields a (multi)graph G^* which is at most $(k-|S'|)$-edge-connected. (For if we delete any $k - |S'| - 1$ edges from G^*, the resulting (multi)graph has $|S'| + 2$ vertices and $|S'| + 1$ edges. But such a (multi)graph is clearly at most 1-edge-connected.)

A vertex cutset in G can be formed from $S' \cup V(M)$ together with one endvertex (appropriately chosen in G) of each edge in a minimum edge cutset in G^*. (Note that this procedure could fail only in the case when $|S'| = 0$ and $k = 1$. However, such failure would indicate that $G - V(M)$ consists of two vertices joined by the single edge in K and hence $|V(G)| = 2m + 2$, contradicting the minimum order condition in the statement of the theorem.) Such a cutset L has at most $2m + |S'| + k - |S'| = 2m + k$ vertices. Since G is $(2m + k + r - 2)$-connected, this yields $2m + k \geqslant 2m + k + r - 2$ or $r \leqslant 2$, a contradiction and the claim is proved.

As a result of the claim, we see that $G - V(M) - S'$ is disconnected and thus $|V(M)| + |S'| \geqslant 2m + k + r - 2$: that is, $|S'| \geqslant k + r - 2$.

Since G is $(2m + k + r - 2)$-connected, each vertex in $V(D)$ has degree (in G'') at least $2m + k + r - 2$. There are k edges between the v_i's, so

$$N \geqslant (|S'| + 2)(2m + k + r - 2) - 2k.$$

We now wish to bound N above. Let us view N from $S' \cup V(M)$. We distinguish two cases.

First suppose $k < r$. Let $u \in V(M) \cup S'$. Suppose first that u is adjacent to both endvertices of no f_i. Then u is adjacent to at most $r - 1$ vertices of D since G is $K_{1,r}$-free. Next suppose, for some j, $1 \leqslant j \leqslant k$, u is adjacent to both ends of some j edges in K. Then, since G is $K_{1,r}$-free, vertex u has at most $2j + (r - j - 1) = j + r - 1 \leqslant k + r - 1$ neighbors in $\{v_1, \ldots, v_{|S'|+2}\}$. Thus in any case, $N \leqslant (2m + |S'|)(k + r - 1)$.

So $(2m + |S'|)(k + r - 1) \geqslant N \geqslant (|S'| + 2)(2m + k + r - 2) - 2k = (|S'| + 2)(2m + (k + r - 1) - 1) - 2k$. Subtracting $|S'|(k + r - 1)$ from both sides, we get

$$2m(k + r - 1) \geqslant 2m|S'| + 4m + 2r - |S'| - 4. \tag{2.1}$$

Now, $|S'| \geqslant k + r - 2$. Substituting this bound for $|S'|$ into the corresponding positive term on the right-hand side of (2.1) one obtains

$$2m(k + r - 1) \geqslant 2m((k + r - 1) - 1) + 4m + 2r - |S'| - 4,$$

i.e.

$$|S'| \geqslant 2m + 2r - 4 \geqslant 2m + k + r - 3, \tag{2.2}$$

since $k < r$. Substituting this new bound for $|S'|$ from (2.2) into the corresponding positive term on the right-hand side of (2.1) one obtains

$$2m(k + r - 1) \geqslant 2m(2m + (k + r - 1) - 2) + 4m + 2r - |S'| - 4,$$

i.e.

$$|S'| \geqslant (2m)^2 + 2r - 4 \geqslant (2m)^2 + k + r - 3, \tag{2.3}$$

again using the assumption that $k<r$. Substituting this new bound for $|S'|$ from (2.3) into the corresponding positive term on the right-hand side of (2.1) one gets

$$2m(k+r-1) \geqslant 2m((2m)^2+(k+r-1)-2)+4m+2r-|S'|-4,$$

i.e.

$$|S'| \geqslant (2m)^3+2r-4 \geqslant (2m)^3+k+r-3. \tag{2.4}$$

Continuing in this way we find, after substituting the bound for $|S'|$ from $(2j)$ into the corresponding positive term on the right-hand side of (2.1), that

$$2m(k+r-1) \geqslant 2m((2m)^j+(k+r-1)-2)+4m+2r-|S'|-4,$$

i.e.

$$|S'| \geqslant (2m)^j+2r-4 \geqslant (2m)^j+k+r-3. \tag{2j+1}$$

Thus, since $m \geqslant 1$, $|S'|$ is unbounded above, contradicting the finiteness of G.

Hence, we may assume that $k \geqslant r$. Remembering that G is $(2m+k+r-2)$-connected and $K_{1,r}$-free and that $r \geqslant 3$, we have

$$\begin{aligned}(|S'|+2m)(2(r-1)) &> (|S'|+2m)(r-1)\\ &\geqslant N\\ &\geqslant (|S'|+2)(2m+k+r-2)-2k\\ &= (|S'|+2)(2m+2(r-1)+(k-r))-2k.\end{aligned}$$

Subtracting $|S'|(2(r-1))$ from both sides of the inequality we get

$$2m(2(r-1)) > |S'|(2m+(k-r))+4m+2r-4.$$

Adding $4m$ to both sides, the inequality becomes

$$\begin{aligned}4mr &> |S'|(2m+(k-r))+8m+2r-4\\ &= 2m|S'|+|S'|(k-r)+8m+2r-4\\ &\geqslant 2m(k+r-2)+|S'|(k-r)+8m+2r-4\\ &\geqslant 2m(2r-2)+|S'|(k-r)+8m+2r-4 \quad (\text{since } k \geqslant r)\end{aligned}$$

i.e.

$$0 > |S'|(k-r)+4m+2r-4 \geqslant 4m+2 \quad (\text{since } r \geqslant 3).$$

The right-hand side of this last inequality is strictly positive and consequently we have reached a contradiction. The proof of the theorem is now complete. □

If $n \leqslant r-1$, then the connectivity hypothesis in the preceding theorem is sharp in the sense that it is easy to construct an even graph G which is $(2m+n+r-3)$-connected and $K_{1,r}$-free but which is not $E(m,n)$. To construct such a G, let H_1 be the complete graph $K_{2m+n+r-3}$ and let H_2 consist of n independent edges and an additional $r-1-n$ isolated vertices. Then let $G=H_1+H_2$. It is easy to verify that G is $(2m+n+r-3)$-connected and $K_{1,r}$-free. Let M be any set of m independent edges in H_1 and N be the n independent edges in H_2. The reader can easily verify that there exists no perfect matching in G containing M and avoiding N. So G is not $E(m,n)$.

If $n \geqslant r$ we do not know if the conclusion of Theorem 2.2 holds if the connectivity hypothesis is reduced by 1.

Sumner [15] proved the following result.

Theorem 2.3. *If $r \geqslant 3$ and G is $(r-1)$-connected, $K_{1,r}$-free and even, then G contains a perfect matching.*

Furthermore, for $r=3$, Sumner [15], and independently Las Vergnas [5], also obtained the following stronger result.

Theorem 2.4. *If G is connected, claw-free and even, then G contains a perfect matching.*

In light of Theorem 2.3 it is tempting to conjecture that Theorem 2.4 can be improved to state that every $(r-2)$-connected, $K_{1,r}$-free even graph contains a perfect matching, when $r \geqslant 4$. But this is false for all $r \geqslant 4$. We now present counterexamples for all such r.

For $r=4$, let Γ_4 be the 10 vertex graph obtained from K_4 by subdividing each edge with a single vertex. For Γ_r, $r \geqslant 5$, we first state and prove the following lemma.

Lemma 2.5. *For all $r \geqslant 5$, there exists a graph G_r which is $(r-2)$-connected, $(r-2)$-regular and bipartite and which has $2(2r-4)=4r-8$ vertices.*

Proof. Our construction is inductive. For $r=5$, let G_5 be $C_6 \times K_2$ (i.e. the hexagonal prism).

Assume that for all r, $5 \leqslant r < k$, we have constructed G_r. Since G_{k-1} is bipartite and regular, by König's Edge-Coloring Theorem [3,4] it must contain a perfect matching M_{k-1}. Let the vertices of G_{k-1} be labelled such that the matching $M_{k-1}=\{b_iw_i \mid 1 \leqslant i \leqslant 2k-6\}$. To construct G_k from G_{k-1} we use four new vertices $U=\{b,b',w,w'\}$, together with the new edges obtained by joining b to $w_1,\ldots,w_{k-3}$, b' to $w_{k-2},\ldots,w_{2k-6}$, w to $b_{k-2},\ldots,b_{2k-6}$, w' to $b_1,\ldots,b_{k-3}$, b to w and b' to w'. The graph G_k is illustrated in Fig. 1.

It is obvious that G_k is bipartite, $(k-2)$-regular and has $2(2k-4)$ vertices.

It remains to show that G_k is $(k-2)$-connected. Let S be a minimum cutset in G_k and assume, to the contrary, that $|S| \leqslant k-3$. Let $S'=S \cap V(G_{k-1})$.

First assume that $G_{k-1}-S'$ is connected. If each $u \in U$ has a neighbor in $G_{k-1}-S'$, then G_k-S is connected, a contradiction. So without loss of generality we may assume that $S'=N_{G_k}(b)=\{w_1,\ldots,w_{k-3}\}$. Hence $S=S'$. But then, G_k-S-b is connected

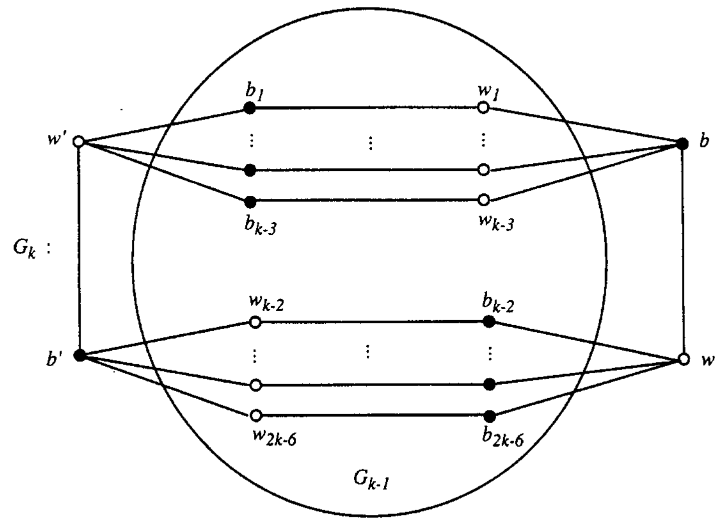

Fig. 1.

and therefore, since b is adjacent to w, $G_k - S$ is connected as well. But this is a contradiction.

So we may assume that $G_{k-1} - S'$ is not connected. But then since G_{k-1} is $(k-3)$-connected and $|S| \leqslant k-3$, we must have $k-3 \leqslant |S'| \leqslant |S| \leqslant k-3$ and hence $S = S'$ and $|S| = k-3$. Moreover, $S \subseteq V(G_{k-1})$. But if H_k is the spanning subgraph of G_k having as its edge set M_{k-1} together with all the edges incident with vertices in U, it is clear that H_k – and hence G_k – cannot be separated by any subset of $V(G_{k-1})$ of size $k-3$. This contradiction establishes the lemma. □

We now proceed to construct Γ_r, $r \geqslant 5$. Let $B \cup W$ be the bipartition of G_r, where B denotes the set of $2r-4$ 'black' vertices and W, the set of $2r-4$ 'white' vertices (see again Fig. 1). Let β_1 and β_2 be two new 'black' vertices. Join β_1 to half the white vertices in G_r and β_2 to the other half. Let the resulting graph be Γ_r. Then $\deg \beta_i = r-2$, for $i = 1,2$ and it follows that Γ_r is $(r-2)$-connected. Moreover Γ_r has maximum degree $r-1$ and hence Γ_r must be $K_{1,r}$-free. On the other hand, Γ_r is an unbalanced bipartite graph and hence has no perfect matching.

3. A result for bipartite graphs

In this section we shall generalize a theorem on extending matchings in regular bipartite graphs first obtained in [1].

Before stating the main result, we present three lemmas.

Lemma 3.1. *Let n and r be non-negative integers. Let G be an r-regular bipartite graph with $r \geqslant n+1$. Then G is $E(0,n)$.*

Proof. Let $f_1,\dots,f_n$ be a set of n independent edges in G. Since G is a regular bipartite graph, by König's Theorem, $E(G)$ may be partitioned into r perfect matchings. Now, $r \geqslant n+1$ so there exists a perfect matching of G which avoids all of the edges $f_1,\dots,f_n$. □

Recall next that the *cyclic-edge-connectivity* of graph G, denoted by $c\lambda(G)$, is the cardinality of any smallest set of edges $L \subseteq E(G)$ such that $G-L$ consists of at least two components each of which contains a cycle.

Lemma 3.2. *If G is an r-regular graph with cyclic connectivity $c\lambda(G) \geqslant r \geqslant 2$, then G is r-edge-connected.*

Proof. Suppose G has a minimal edge cut L, with $|L|<r$. Then at least one (of the two) components of $G-L$ is a tree. Let this component be T.

If $|V(T)|=1$, then the r-regularity assumption is contradicted. So $|V(T)| \geqslant 2$ and hence tree T has at least two endvertices. Each of these two endvertices of T is incident with exactly $r-1$ edges of L. Hence $|L| \geqslant 2(r-1)$. But $2(r-1) \geqslant r$ since $r \geqslant 2$. So $|L| \geqslant r$, a contradiction. □

Lemma 3.3. *Let F be a forest with no isolated vertices. Suppose the bipartition of $V(F)$ is $A \cup B$, where $|A|=a$, $|B|=b$ and $a>b$. Then at least one component of F is tree with at least two endvertices in A.*

We are now prepared to state and prove the main result of this section.

Theorem 3.4. *Let m,n and r be non-negative integers with $r \geqslant \max\{2n+1, m+2\}$. Let G be an r-regular bipartite graph with $|V(G)| \geqslant 2m+2n+2$ and*

$$c\lambda(G) \geqslant \begin{cases} 0 & \text{when } m=0, \\ (m-1)r+2n+1 & \text{for all } m \geqslant 1. \end{cases}$$

Then G is $E(m,n)$.

Proof. Note that $r \geqslant 2$. Moreover, if $r=2$, then $m=0$ and $n=0$. But trivially, every 2-regular bipartite graph G contains a perfect matching, i.e. G is $E(0,0)$. In consequence, we shall assume henceforth that $r \geqslant 3$.

When $m=0$, since $r \geqslant 2n+1 \geqslant n+1$, the result follows immediately as in the proof of Lemma 3.1. (In fact, Lemma 3.1 is stronger in that it requires a less stringent lower bound on r.)

When $m=1$, since $c_\lambda(G)\geqslant 2n+1$ and $r\geqslant 2n+1$, we claim that G is $(2n+1)$-edge-connected. To see this let L be a minimum edge cut in G. If L is a *cyclic* edge cut, we are done. So suppose $G-L$ has a component T which is a tree. If T is a single vertex, $|L|=r\geqslant 2n+1$, by the definition of r and again we are done. Hence, suppose that T contains at least two vertices and therefore at least two endvertices. It follows that $|L|\geqslant 2(r-1)>r\geqslant 2n+1$, since $r\geqslant 3$. Thus by Theorem 3.1 of Aldred et al. [1], G is $E(1,n)$.

For the remainder of the proof, we shall assume that $m\geqslant 2$. We shall proceed by induction on n, noting that when $n=0$, the result follows immediately from Theorem 2.2 of Plummer [9]. Assume that for all values of $n<k$, the theorem holds and suppose that the theorem fails for $n=k$, i.e., let G be an r-regular bipartite graph with $|V(G)|\geqslant 2m+2k+2$, $r\geqslant \max\{2k+1,m+2\}$ and $c_\lambda(G)\geqslant (m-1)r+2k+1$ and suppose that G is not $E(m,k)$. So there exist two sets of independent edges, $M=\{e_1,\dots,e_m\}$, $K=\{f_1,\dots,f_k\}$ and $M\cap K=\emptyset$, such that $G'=G-V(M)-K$ contains no perfect matching. Let $A\cup B$ be the bipartition of $V(G)$ and for $i=1,\dots,m$, let $e_i=a_ib_i$, where $a_i\in A$ and $b_i\in B$. Then by the bipartite matching theorem of Philip Hall, we may assume, without loss of generality, that there exists a vertex set $A_1\subseteq A$ with neighborhood $B_1\subseteq B$ in G' such that $|A_1|>|B_1|$. Let A_2 be the set consisting of $\{a_1,a_2,\dots,a_m\}$ and let B_2 be the set consisting of $\{b_1,b_2,\dots,b_m\}$. Finally, let $A_0=A-(A_1\cup A_2)$ and let $B_0=B-(B_1\cup B_2)$.

By the choice of k, G is $E(m,k-1)$ and hence $G'\cup f_i$ contains a perfect matching for each $i=1,\dots,k$. Hence, each f_i must join a vertex in A_1 to a vertex of B_0. Furthermore, $|A_1|=|B_1|+1$ and hence $|B_0|=|A_0|+1$. We denote by G_i the subgraph of G induced by $A_i\cup B_i, i=0,1,2$. We note that, (i) since the degree of every vertex in A_1 is at least $m+2$, (ii) since K is a matching, and (iii) since $|B_2|=m$, it follows that the set B_1 is not empty. Furthermore, by a symmetric argument, since $B_0\neq\emptyset, A_0\neq\emptyset$.

For the rest of this proof, we adopt the following terminology (see Fig. 2):

- $q=$ the number of edges from A_1 to B_2,
- $n_0=$ the number of edges from B_0 to A_2,
- $n_1=$ the number of edges from A_0 to B_1,
- $n_2=$ the number of edges from A_2 to B_1,
- $n_3=$ the number of edges from B_2 to A_0, and
- $n_4=$ the number of edges in G_2.

Counting edges in G_1, we have $|A_1|r-q-k=|B_1|r-n_1-n_2$, or

$$|A_1|r=|B_1|r-n_1-n_2+q+k. \tag{3.1}$$

Also $|A_1|=|B_1|+1$, so

$$|A_1|r=(|B_1|+1)r. \tag{3.2}$$

Combining (3.1) and (3.2), we get $(|B_1|+1)r=|A_1|r=|B_1|r-n_1-n_2+q+k$, or

$$k+q=n_1+n_2+r. \tag{3.3}$$

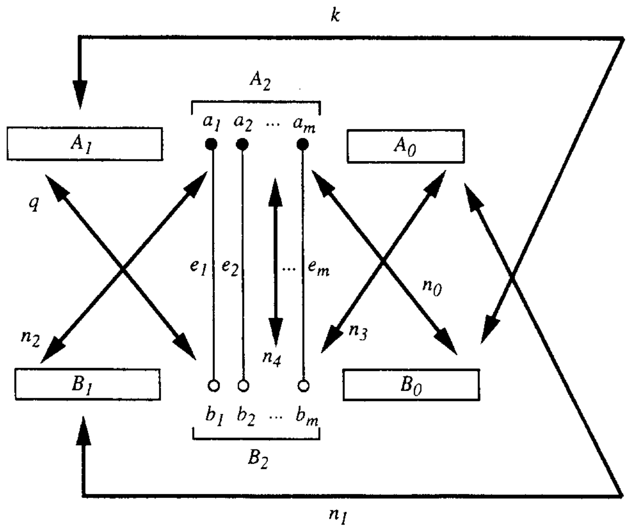

Fig. 2.

Claim 1. *If G_0 is acyclic, then $|A_0| + |B_0| \leqslant 2m + 1$.*

For if G_0 is a forest, $|V(G_0)| \geqslant |E(G_0)| + 1$, and so

$$\begin{aligned} 2(|V(G_0)|) &= 2(|A_0| + |B_0|) \\ &\geqslant 2|E(G_0)| + 2 \\ &= \left(\sum_{v \in V(G_0)} \deg_{G_0} v \right) + 2 \\ &= |A_0|r - n_1 - n_3 + |B_0|r - n_0 - k + 2 \end{aligned}$$

or

$$(r-2)(|A_0| + |B_0|) \leqslant n_0 + n_1 + n_3 + k - 2. \tag{3.4}$$

Now counting edges out of A_2 in G, we also have

$$rm = n_0 + n_2 + n_4 \geqslant n_0 + n_2 + m$$

so

$$n_0 + n_2 \leqslant (r-1)m$$

and hence

$$n_0 \leqslant (r-1)m - n_2, \tag{3.5}$$

and similarly, counting edges out of B_2 in G, we also have

$$n_3 \leqslant (r-1)m - q. \tag{3.6}$$

Substituting (3.5), (3.3) and (3.6) into (3.4), we get

$$\begin{aligned}(r-2)(|A_0|+|B_0|) &\leqslant k+(r-1)m-n_2+n_1+n_3-2\\ &= k+(r-1)m+n_1+n_2-2n_2+n_3-2\\ &= k+(r-1)m+k+q-r-2n_2+n_3-2\\ &\leqslant 2k+(r-1)m+q-r-2n_2+(r-1)m-q-2\\ &= 2k+2(r-1)m-r-2n_2-2\\ &\leqslant 2k+2(r-1)m-r-2.\end{aligned}$$

Noting that $r-2>0$ and dividing both sides by $r-2$, we obtain

$$\begin{aligned}(|A_0|+|B_0|) &\leqslant \frac{2k+2(r-1)m}{r-2}-\frac{r+2}{r-2}\\ &< \frac{2m(r-1)}{r-2}+\frac{2k}{r-2}-\frac{r-1}{r-2}\\ &= (2m-1)\left(\frac{r-1}{r-2}\right)+\frac{2k}{r-2}\\ &= (2m-1)\left(1+\frac{1}{r-2}\right)+\frac{2k}{r-2}\\ &= (2m-1)+\frac{2m-1}{r-2}+\frac{2k}{r-2}\\ &\leqslant (2m-1)+2-\frac{1}{m}+1+\frac{1}{2k-1}\\ &= 2m+2+\left(\frac{1}{2k-1}-\frac{1}{m}\right).\end{aligned}$$

Now $1/(2k-1)\leqslant 1$ for all $k\geqslant 0$, so $1/(2k-1)-1/m<1$. So $|A_0|+|B_0|<2m+2+1=2m+3$. But $|A_0|+|B_0|$ is odd, so $|A_0|+|B_0|\leqslant 2m+1$, and thus Claim 1 is proved.

Claim 2. $n_1+n_2+n_3+n_4=r(m-1)+k$.

To see this, note that

$$\begin{aligned}n_1+n_2+n_3+n_4 &= n_1+n_2+n_3+(rm-q-n_3)\\ &= k+q-r+n_3+(rm-q-n_3)\\ &= r(m-1)+k.\end{aligned}$$

We now define $H_0=G[A_0\cup B_0\cup A_2]$ and $H_1=G[A_1\cup B_1\cup B_2]$.

Claim 3. *Subgraph H_1 contains a cycle.*

Suppose not. Then H_1 is a forest and hence so is G_1. If G_1 contains an isolate u, then by the definition of B_1, $u\in A_1$. But G is r-regular and $r\geqslant m+2$, so u must have at least two neighbors in B_0, a contradiction, since all edges between A_1 and B_0 belong to the matching K. Since $|A_1|>|B_1|$, by Lemma 3.3 it follows that one of the components of G_1 is a tree T_1 with at least two endvertices in A_1. Again, since G is r-regular and $r\geqslant m+2$, every endvertex of T_1 is adjacent (in G) to every vertex in B_2 and hence, since $m\geqslant 1$, H_1 contains a cycle; a contradiction.

Claim 4. *H_0 contains a cycle.*

Suppose not. Then H_0 is a forest and hence if $|B_0|=l$ (and thus $|A_0|=l-1$), then

$$\begin{aligned}|V(H_0)| &= |A_0|+|B_0|+m\\ &= 2l-1+m\\ &\geqslant |E(H_0))|+1\\ &= (m+l-1)r-(n_1+n_2+n_3+n_4)+1\\ &= n(m+l-1)r-(r(m-1)+k)+1\\ &= lr-k+1.\end{aligned}$$

So, in particular, $lr-k+1\leqslant 2l-1+m$ and hence

$$lr\leqslant 2l+m+k-2. \tag{3.7}$$

Hence $l(m+2)\leqslant lr\leqslant 2l+m+k-2$ and therefore

$$lm\leqslant m+k-2. \tag{3.8}$$

By Claim 1, $2l-1\leqslant 2m+1$ and hence

$$l\leqslant m+1. \tag{3.9}$$

On the other hand, since the edges in K are independent, $l\geqslant k$, and so by (3.8)

$$lm\leqslant m+l-2. \tag{3.10}$$

Substituting (3.9) into (3.10) we obtain $lm\leqslant m+(m+1)-2=2m-1$. But this is a contradiction since $l\geqslant 2$, and hence Claim 4 is proved.

But then H_0 and H_1 each contain a cycle and are separated by an edge cut of size $n_1+n_2+n_3+n_4+k=r(m-1)+2k<r(m-1)+2k+1$, contradicting the hypothesis on $c_\lambda G$. $\square$

We note that, in the special case when $m=1$ and $r=2n+1$, there exist $(2n+1)$-regular bipartite graphs having cyclic connectivity $2n+1$, but which are not $E(2,n)$ as well as other such graphs which are not $E(1,n+1)$. For examples of both types, the reader is referred to [1].

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